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THE CONNECTION BETWEEN PARTIAL DIFFERENTIAL EQUATIONS SOLUBLE BY INVERSE SCATTERING AND ORDINARY DIFFERENTIAL EQUATIONS OF PAINLEVÉ TYPE

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ABSTRACT (15) DAAG29-80-C-0041

A completely integrable partial differential equation is one which has a Lax representation, or, more precisely, can be solved via a linear integral equation of Gel'fand-Levitan type, the classic example being the Korteweg-de Vries equation. An ordinary differential equation is of Painlevé type if the only singularities of its solutions in the complex plane are poles. It is shown that, under certain restrictions, if G is an analytic, regular symmetry group of a completely integrable partial differential equation, then the reduced ordinary differential equation for the G-invariant solutions is necessarily of Painlevé type. This gives a useful necessary condition for complete integrability, which is applied to investigate the integrability of certain generalizations of the Korteweg-de Vries equation, Klein-Gordon equations, some model nonlinear wave equations of Whitham and Benjamin, and the BBM equation.

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Key Words: Completely integrable partial differential equations, Inverse scattering method, Gel'fand-Levitan equation, KdV equation, Klein-Gordon equations, Nonlinear Schrödinger equation, Similarity solutions, Painlevé transcendents

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SIGNIFICANCE AND EXPLANATION

Several very important nonlinear partial differential equations can be solved by the method known as "inverse scattering". This in effect reduces the solution of the nonlinear equation to that of a linear integral equation, and it is obviously desirable, given any nonlinear partial differential equation, to determine whether its solution is amenable to this technique. Hitherto this determination has been largely a matter of chance, but the present paper gives a relatively simple systematic test. One looks at the ordinary differential equations satisfied by similarity solutions of the nonlinear equation. If these ordinary differential equations are not "of Painlevé type", i.e. if they possess solutions having singularities other than poles, then the nonlinear equation is not soluble by inverse scattering.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE CONNECTION BETWEEN PARTIAL DIFFERENTIAL EQUATIONS SOLUBLE BY INVERSE SCATTERING AND ORDINARY DIFFERENTIAL EQUATIONS OF PAINLEVÉ TYPE

J. B. McLeod and P. J. Olver

1. Introduction

The solution of certain nonlinear partial differential equations by inverse scattering techniques has been the subject of considerable interest in recent years. This technique dates back to a fundamental observation of P. Lax, [26], that some puzzling earlier results of Miura, Gardner and Kruskal, [33], on the relationship between the Korteweg-de Vries equation and the eigenvalue problem for Hill's equation, could be placed in an extremely lucid and general theoretical framework. Lax's basic idea was that if a partial differential equation could be cast into the form

$$\frac{dL}{dt} = [B,L] = BL - LB , \tag{1.1}$$

where L and B are linear (differential) operators on some Hilbert space with B skew-adjoint, then the eigenvalues of L would be independent of time t. We will call (1.1) a <u>Lax representation</u> of the given partial differential equation, and a system of partial differential equations which can be so represented is often called <u>completely integrable</u>, although, as will be seen later in this introduction, we shall in this paper find it convenient to use a somewhat different definition of complete integrability. (The terminology stems from the interpretation of the KdV equation as a completely integrable Hamiltonian system, as discovered by Gardner, [16], and developed in great detail by McKean, van Moerbeke and Trubowitz, [30], [31].)

For the KdV equation, which is

$$u_t + 6uu_x + u_{xxx} = 0 , \tag{1.2}$$

the operators appearing in the Lax representation are

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$$L = -D^2 - u ,$$

$$B = -(4D^3 + 3(Du + uD)) ,$$

where $D = d/dx$. The problem of finding the potential u from the spectral data of L, known as the "inverse scattering problem", was solved by Gel'fand and Levitan, [19]. (See also [14], [42] for comprehensive introductions.) Their method involves the solution of a certain linear integral equation, the <u>Gel'fand-Levitan equation</u>, which in general is of the following form:

$$K(x,y) + F(x,y) + \int_x^\infty K(x,z)H(z,y)dz = 0 . \tag{1.3}$$

(Technically, (1.3) is the Marchenko form, [29], of the Gel'fand-Levitan equation.) Here F and H are constructed from the relevant spectral data of L. Once K has been found, the potential u is recovered from the values of K on the diagonal $x = y$. (In the KdV case, $u = 2D[K(x,x)]$.) Thus the basic technique for solving the initial value problem for a completely integrable system of partial differential equations consists of the following steps:

1) Given the initial data $u(x,0)$, determine the appropriate spectral data of the operator L at time $t = 0$.

2) Find the time evolution of the spectral data, and hence of the kernel functions F and H used in the Gel'fand-Levitan equation.

3) Solve the Gel'fand-Levitan equation, regarding t as a parameter, and thus recover the solution u of the original system.

The inverse scattering technique outlined above has been applied to a number of physically relevant partial differential equations. Of particular interest is the work of Zakharov, Manakov and Shabat, [48]-[53], on the nonlinear Schrödinger and other physically interesting equations, and of Ablowitz, Kaup, Newell and Segur on a general class of 2×2 matrix systems, [2], and the three-wave interaction equations, [24]. In all cases the appropriate Gel'fand-Levitan equation assumes the form (1.3), although F, H and K

may be matrix-valued functions. The formula for recovering the solution u from the kernel K varies from case to case.

The one notable drawback in this elaborate theory is that for a given system of partial differential equations there has not to date been any systematic method of determining whether or not it has a Lax representation and, if so, how to find the operators B and L. Previous work has relied either on inspired guesswork, or else on fixing the form of the operators B and L and seeing what systems of partial differential equations result. In particular, recent work of Gel'fand and Dikii, [17], [18], which lists several other references, has used abstract differential-algebraic methods to give a general classification of pairs of differential operators B and L such that (1.1) corresponds to a bona fide system of partial differential equations. This approach, however, while providing a large number of completely integrable systems, is not yet able to answer the above question of whether a given system is completely integrable.

The basis of the present paper is an observation of Ablowitz and Segur, [5], that the equations for the group-invariant (self-similar) solutions of known examples of completely integrable equations turn out to be ordinary differential equations studied extensively by Painlevé and his students around the turn of the century, [10], [38]. (See also [22], [23] for general accounts of the subject and further references.) These equations are characterized by the property that all their solutions in the finite complex plane possess only poles as singularities, and hereafter we will refer to an ordinary differential equation with this property as an equation of <u>Painlevé type</u>. (Painlevé allowed also fixed singularities of an arbitrary type, but we will not.) Hastings and McLeod, [21], exploited this relationship to solve a nonlinear connection problem for the second Painlevé transcendent, and conjectured that the above relationship was not fortuitous:

<u>Conjecture</u>. <u>If a system of partial differential equations is completely integrable, and</u> G <u>is a symmetry group of this system, then the reduced system of ordinary differential equations for the</u> G<u>-invariant solutions is of Painlevé type</u>.

This conjecture, if true, would provide a powerful necessary condition to test for complete integrability. Here we will prove a somewhat weakened version of the conjecture,

which nevertheless proves useful in several applications. There are two restrictions. First, if, in the Lax operator L, some combination of the solution u and its spatial derivatives occurs, say $Q(u)$, then it is this combination (or combinations) that must have only poles as singularities. For instance, if $L = D^2 + u_x$, then only u_x is required to have poles, and thus we may allow logarithmic branch points as singularities of the solutions of the reduced ordinary differential equations. Usually we will assume that Q is a linear combination of u and its spatial derivatives, calling this case <u>linearly completely integrable</u>. Secondly, the same combination Q must satisfy certain preconditions for the inverse scattering formalism to go through; this means that, when restricted to the real axis, Q either is periodic or satisfies decay conditions at $x = \pm\infty$, which implies corresponding restrictions on the solutions u that can be considered. It is only for such solutions that $Q(u)$ must be meromorphic. If a system of ordinary differential equations has the property that, for such solutions u, the combination $Q(u)$ is meromorphic, we say that the system is of <u>restricted Painlevé type</u> relative to Q. Our basic result, in rough form, replaces "Painlevé type" by "restricted Painlevé type" in the above conjecture.

The first requirement for stating and proving a precise form of the conjecture is to define what is meant by a partial differential equation being completely integrable. Rather than take the Lax representation as our starting point, we shall adopt the more practical view of the Gel'fand-Levitan equation being of primary importance. Thus a completely integrable system of partial differential equations is defined as one whose solutions are found by solving a linear integral equation of special type, cf. Definition 2.1 below. This viewpoint is necessitated by the fact that the formal Lax representation theory of Gel'fand and Dikii has not reached the point of stating an analogue of the result of Gel'fand and Levitan for an arbitrary scattering operator L, although some formal progress in this direction has been made in [28], [53]. Since in all examples known to the authors the Gel'fand-Levitan equation is always of the form (1.3), it might seem reasonable that this should be true in general, but Application V in Section 4 below indicates that it may be necessary to restrict L at least to being of prime order.

The main tool in our proof is a theorem of Steinberg, [43], which states that if $T(z)$ is an analytic family of compact operators in a Banach space, then $(I - T(z))^{-1}$, provided this inverse exists for at least one value of z, is a meromorphic family of operators. Under appropriate assumptions on the initial data of our completely integrable system to ensure that the functions F and H in the Gel'fand-Levitan equation satisfy certain analyticity criteria, we can conclude from Steinberg's result that Q must be a meromorphic function of (x,t). Now suppose that G is a one-parameter, analytic, regular local group of transformations acting on the space of independent and dependent variables which leaves the set of solutions of the system of partial differential equations invariant. Then the G-invariant (self-similar) solutions can all be found by integrating a system of ordinary differential equations on the quotient manifold whose points correspond to the orbits of G. (See Section 3 for details.) The analyticity of G implies that for any G-invariant solution whose initial data satisfies the inverse scattering assumptions, the function Q on the quotient manifold can have only poles for singularities. In other words, the reduced system of ordinary differential equations must be of restricted Painlevé type relative to Q. This completes the outline of the proof of our main theorem; precise statements and proofs will appear in Sections 2 and 3.

In Section 4 we discuss some applications of this result. First we show that the generalized KdV equation

$$u_t + u^p u_x + u_{xxx} = 0$$

can be linearly completely integrable only if $p = 0, 1,$ or 2. These exceptional cases correspond to the Airy equation in moving coordinates, the KdV and the modified KdV equations, which are well known to be completely integrable. Secondly we consider a nonlinear Klein-Gordon equation in characteristic coordinates

$$u_{xt} = f'(u) \; . \tag{1.4}$$

It is shown that if $f(u)$ is a rational function, real for real u and with two consecutive zeros, simple or double, on the real axis, and if (1.4) is linearly completely integrable, then f is a polynomial of degree at most 4. Further, if $f(u)$ is a linear

combination of exponentials $e^{\alpha_j u}$ with the α_j all rational multiples of some complex number α, again real for real u and with two consecutive simple or double zeros, and if (1.4) is linearly completely integrable, then

$$f(u) = c_2 e^{2\beta u} + c_1 e^{\beta u} + c_0 + c_{-1} e^{-\beta u} + c_{-2} e^{-2\beta u} ,$$

for some number β. The next application shows that certain nonlinear model wave equations considered by Benjamin, Bona and Mahony, [7], and Whitham, [46], cannot be linearly completely integrable. The last example deals with the BBM equation, [7],

$$u_t + uu_x - u_{xxt} = 0 . \tag{1.5}$$

Although this cannot be treated rigorously by the methods of the present paper, we show that if the full conjecture were true, then (1.5) could not be linearly completely integrable. From these results, it can be seen that our criterion for complete integrability is a powerful preliminary test when considering whether or not a system of partial differential equations can be integrated by inverse scattering techniques.

A recent preprint of Ablowitz, Ramani and Segur, [3], also considers the above conjecture. They prove a similar result, although they place a rather severe restriction on the form of the functions occurring in the Gel'fand-Levitan equation for the group-invariant solutions which is not sufficiently justified. Also, the only groups they consider are groups of scale transformations; the more general groups that we consider allow a much wider range of applications.

2. Analyticity Properties of Completely Integrable Differential Equations

Consider a system of partial differential equations

$$\Delta(t,x,u) = 0 , \tag{2.1}$$

where $x,t \in \mathbf{R}$ and $u = (u^1,\ldots,u^m) \in \mathbf{R}^m$ is a vector-valued function. We assume that the initial value problem of (2.1) with

$$u(x,0) = f(x) \tag{2.2}$$

is well posed for f in some Banach space $\mathcal{B}$ of functions, so that for t sufficiently small, there is a unique solution $u(x,t)$ of (2.1-2). In practice $\mathcal{B}$ is either a space of functions decreasing sufficiently rapidly at $\pm\infty$ or a space of periodic functions. Usually the presence of appropriate conservation laws will ensure that the solutions are actually global in t, but this will not be assumed a priori. The first task is to make precise what is meant by (2.1) being completely integrable. Rather than use the usual Lax representation of the equation, we will assume a more practical outlook and take the Gel'fand-Levitan integral equation as our starting point. As the examples demonstrate, in all known cases there is such an integral equation for solution of the inverse scattering problem for the operator L in the Lax representation (1.1).

<u>Definition 2.1</u>. A system of partial differential equations is <u>completely integrable relative to</u> $\Omega(u)$ <u>in the Banach space</u> $\mathcal{B}$ if there is a linear matrix integral equation of the form

$$K(x,y;t) + F(x,y;t) + \int_x^\infty K(x,z;t)H(z,y;t)dz = 0 , \tag{2.3}$$

called the <u>Gel'fand-Levitan equation</u>, satisfying the following properties:

i) F, H, K are $N \times N$ matrices of functions;

ii) F and H are uniquely determined by the initial data (2.2);

iii) for initial data in $\mathcal{B}$, and for all real x, y, all complex ε, and t in some domain Ω in $\mathbf{C}$, the functions $F(x - \varepsilon t, y - \varepsilon t;t)$ and $H(x - \varepsilon t, y - \varepsilon t;t)$ are analytic in ε, t, and there is a Banach space $\mathcal{B}^*$ (not necessarily the same as $\mathcal{B}$) for which $F(x - \varepsilon t, y - \varepsilon t;t) \in \mathcal{B}^*$ as a function of y and the operator

$$T(x,t)f(y) = \int_x^\infty f(z)H(z - \varepsilon t, y - \varepsilon t; t)\,dz$$

is a compact operator in $\mathcal{B}^*$;

iv) the Gel'fand-Levitan equation has a unique solution (in $\mathcal{B}^*$) for all x and at least one t in Ω;

v) the solution u of the system (2.1-2) can be recovered from the solution K of the Gel'fand-Levitan equation via a relation of the form

$$Q[u(x,t)] = P[K(x,x,t)], \tag{2.4}$$

where Q is some function of u and its spatial derivatives and P is a polynomial in K and its spatial derivatives.

Thus to recover the solution u of a completely integrable system of partial differential equations, we must solve the Gel'fand-Levitan equation for K, and then solve the differential equation (2.4) for u. In practical examples, Q is a linear combination of the spatial derivatives of u, and in this case the system will be called <u>linearly completely integrable</u>. It should also be remarked that the requirement that iii) hold for all complex ε can certainly be relaxed, although there seems little practical point in doing so, and that the domain Ω will customarily include the origin or at least have the origin on its boundary (it might, as in the example of the KdV equation below, be a sector of a circle centre the origin).

We now illustrate the definition with two well known examples.

<u>Example 2.2</u>: <u>The Korteweg-de Vries Equation</u>

This is the original example of the use of inverse scattering techniques, [26], [32]. The equation is

$$u_t + 6uu_x + u_{xxx} = 0\ , \tag{2.5}$$

and has a Lax representation with operators

$$L = -D^2 - u\ , \tag{2.6}$$

$$B = -\{4D^3 + 3(Du + uD)\}\ ,$$

where $D = d/dx$. In the case that these operators act on a dense subspace of $L^2(\mathbf{R})$ the appropriate spectral data of L consists of

i) the eigenvalues $\lambda_1 = -k_1^2, \ldots, \lambda_n = -k_n^2$,

ii) the associated norming constants, $c_1, \ldots, c_n$, defined so that if φ_j is the eigenfunction associated with λ_j satisfying $\varphi_j(x) \sim e^{-k_j x}$, $x \to +\infty$, then $c_j^{-1} = \int_{-\infty}^{\infty} \varphi_j^2 dx$,

iii) the reflection coefficient $R(k)$ for k real and positive, $k^2 = \lambda$, defined so that there exists a solution φ of $L\varphi = \lambda\varphi$ which satisfies the boundary conditions

$$\varphi(x) \sim \begin{cases} e^{-ikx} + R(k)e^{ikx}, & x \to +\infty , \\ T(k)e^{-ikx}, & x \to -\infty . \end{cases}$$

The time evolution of the spectral data is determined by

$$\begin{aligned} \lambda_j(t) &= \lambda_j(0) , \\ c_j(t) &= c_j(0) \exp[8k_j^3 t] , \\ R(k,t) &= R(k,0) \exp[8ik^3 t] . \end{aligned} \tag{2.7}$$

The Gel'fand-Levitan equation takes the form

$$K(x,y;t) + F(x + y;t) + \int_x^{\infty} K(x,z;t)F(z + y,t)dz = 0 , \tag{2.8}$$

and the solution of the KdV equation is recovered via the simple formula

$$u(x,t) = 2 \frac{d}{dx} K(x,x;t) . \tag{2.9}$$

The kernel F of the Gel'fand-Levitan equation is given by

$$F(x,t) = \sum_{j=1}^{n} c_j(t)\exp[-k_j x] + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k,t)e^{ikx}dk , \tag{2.10}$$

and this solution is valid provided the initial data $u(x,0) = f(x)$ satisfies

$$\int_{-\infty}^{\infty} (1 + |x|)|f(x)|dx < \infty . \tag{2.11}$$

The uniqueness of the solution of (2.8) in the KdV case is a standard result, and the only item remaining to be checked is Definition 2.1 in condition iii). So far as analyticity is concerned, the only part of F that could fail to be analytic is that corresponding to the continuous spectrum of L:

$$F_c(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ikx}dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,0)\exp[8ik^3t + ikx]dk \ .$$

If we take any reasonable space of initial data for $\mathcal{B}$, for example that given by (2.11), then $R(k,0)$ can be extended analytically into the upper half of the k-plane, and $|R(k,0)/k^2|$ is bounded as $|k| \to \infty$. (The function R is closely related to the spectral density function m of Titchmarsh, and the analyticity and estimates can be obtained by suitably translating the results in [44, Chapter V].) If therefore we write

$$F_c(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\}R(k,0)\exp[8ik^3t + ikx]dk = F_2 + F_1, \quad \text{say} ,$$

and consider sufficiently F_1, then, if t is real and positive, we can deform the integral from $(0,\infty)$ to $(0,\alpha\infty)$, for any α with $0 < \arg\alpha < \frac{1}{3}\pi$. We can now increase $\arg t$, but the range for α becomes $0 < \arg\alpha < \frac{1}{3}(\pi - \arg t)$. Nonetheless this does allow us to define $F_1(x,t)$ as an analytic function of t for $0 < \arg t < \pi$. (It is also an analytic function of x since for large k the term k^3t dominates kx.) If we decrease $\arg t$, the range for α becomes $-\frac{1}{3}\arg t < \arg\alpha < \frac{1}{3}\pi$, which allows us to define $F_1(x,t)$ as an analytic function of t for $-\pi < \arg t < 0$, and so in fact in the whole complex plane cut along the negative axis. Similar remarks apply to F_2.

Further, by using the deformed contours and integrating by parts (integrating e^{ikx} and differentiating the remainder), we see that $F \in \mathcal{B}$, the Banach space defined by

(2.11), and that the operator T is compact in $\mathcal{B}$, although $\mathcal{B}$ is certainly not the only possible choice for $\mathcal{B}^*$.

Example 2.3: The AKNS Systems

A generalized eigenvalue problem considered by Ablowitz, Kaup, Newell and Segur, [1], [2], continuing work of Zakharov and Shabat, [51], assumes the following form:

$$\begin{aligned} v_x &= -i\zeta v + q(x)w \ , \\ w_x &= i\zeta w + r(x)v \ , \end{aligned} \tag{2.12}$$

where ζ corresponds to the eigenvalue and $q(x), r(x) \to 0$ as $|x| \to \infty$. If we include a "time" dependence and suppose that the potentials $q(x,t)$, $r(x,t)$ evolve according to some evolution equation in such a way that ζ is independent of t and, say,

$$\begin{aligned} v_t &= \alpha(x,t,\zeta)v + \beta(x,t,\zeta)w \ , \\ w_t &= \gamma(x,t,\zeta)v - \alpha(x,t,\zeta)w \ , \end{aligned} \tag{2.13}$$

then cross-differentiation between (2.12) and (2.13) leads to

$$\begin{aligned} \alpha_x &= -r\beta + q\gamma \ , \\ \beta_x + 2i\zeta\beta &= q_t - 2q\alpha \ , \\ \gamma_x - 2i\zeta\gamma &= r_t + 2r\alpha \ . \end{aligned} \tag{2.14}$$

The equations (2.14) can also be reached via a Lax representation (1.1), if we take

$$L = \begin{pmatrix} iD & -iq \\ ir & -iD \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} .$$

The insistence that (1.1) should be valid when operating on solutions $\binom{v}{w}$ of (2.12) leads again to (2.14), and conversely.

By judicious choice of the functions α, β, γ, the AKNS system (2.14) gives rise to a large variety of physically interesting equations, including the KdV, modified KdV, sine-Gordon, nonlinear Schrödinger, and others. For instance,

$$q = u, \quad r = \bar{u} ,$$
$$\alpha = -2i\zeta^2 - i|u|^2 ,$$
$$\beta = 2\zeta u + iu_x ,$$
$$\gamma = 2\zeta\bar{u} - i\bar{u}_x ,$$

leads to the nonlinear Schrödinger equation

$$iu_t = -u_{xx} + 2u|u|^2 ,$$

whereas

$$q = -\frac{1}{2}u_x, \quad r = \frac{1}{2}u_x ,$$
$$\alpha = \frac{i}{4\zeta}\cos u ,$$
$$\beta = \gamma = \frac{i}{4\zeta}\sin u$$

leads to the sine-Gordon equation

$$u_{xt} = \sin u .$$

In fact, a system of AKNS type with any given dispersion relation can be found!

The spectral data for (2.12) is constructed as follows. For ζ real, consider the solutions satisfying the following asymptotic boundary conditions:

$$\varphi \sim \binom{1}{0}e^{-i\zeta x} , \qquad \tilde{\varphi} \sim \binom{0}{-1}e^{i\zeta x} , \qquad x \to -\infty ;$$

$$\psi \sim \binom{0}{1}e^{i\zeta x} , \qquad \tilde{\psi} \sim \binom{1}{0}e^{-i\zeta x} , \qquad x \to +\infty .$$

Then

$$\varphi = a(\zeta,t)\tilde{\psi} + b(\zeta,t)\psi ,$$
$$\tilde{\varphi} = -\tilde{a}(\zeta,t)\psi + \tilde{b}(\zeta,t)\tilde{\psi}$$

for some $a, b, \tilde{a}, \tilde{b}$. If $(1 + |x|)q, (1 + |x|)r \in L^1(\mathbb{R})$, then a (respectively $\tilde{a}$) can, as functions of ζ, be analytically extended to the upper (lower) half plane, and their zeros correspond to the eigenvalues of (2.12). Let $F = F_c + F_d$, where

$$F_c(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\zeta,t)}{a(\zeta,t)}\, e^{i\zeta x} d\zeta ,$$

which corresponds to the continuous spectrum, and

$$F_d(x,t) = -i \sum_{n=1}^{N} c_n e^{i\zeta_n x} ,$$

which corresponds to the discrete spectrum consisting of the zeros, ζ_j, of $a(\zeta)$ in the upper half plane (for convenience assumed simple) and appropriate norming constants c_j. Similarly construct $\tilde{F} = \tilde{F}_c + \tilde{F}_d$ from $\tilde{a}$ and $\tilde{b}$. Then if

$$\mathcal{F} = \begin{pmatrix} 0 & -\tilde{F} \\ F & 0 \end{pmatrix} ,$$

the Gel'fand-Levitan equation takes the form (with t-dependence suppressed)

$$K(x,y) + \mathcal{F}(x + y) + \int_x^{\infty} K(x,z)\mathcal{F}(z + y)dz = 0 ,$$

where

$$K = \begin{pmatrix} \tilde{K}_1 & K_1 \\ \tilde{K}_2 & K_2 \end{pmatrix} .$$

The potentials q and r are recovered via

$$q(x) = -2K_1(x,x), \quad r(x) = -2\tilde{K}_2(x,x) .$$

(Note that the sine-Gordon case gives an example in which the operator Q in (2.4) is nontrivial.) Finally, it remains to discuss the time evolution of the spectral data. Under the assumption that

$$\alpha(x,t,\zeta) \to \alpha_o(\zeta), \quad \beta(x,t,\zeta) \to 0, \quad \gamma(x,t,\zeta) \to 0$$

as $|x| \to \infty$, it can be shown that

$$\begin{aligned} a(\zeta,t) &= a(\zeta,0) , \\ b(\zeta,t) &= b(\zeta,0)\exp[-2\alpha_o(\zeta)t] , \\ \zeta_j(t) &= \zeta_j(0) , \\ c_j(t) &= c_j(0)\exp[-2\alpha_o(\zeta_j)t] . \end{aligned}$$

If, in addition, $|\alpha_0(\zeta)/\zeta| \to \infty$ as $|\zeta| \to \infty$, then arguments similar to those used in the previous example show that, for suitable initial data, $F(x,t)$ is analytic in t for all x, and also analytic in x. A large class of AKNS systems satisfies these additional assumptions.

We now investigate the properties of the solutions of a general integral equation of Gel'fand-Levitan type. Our main tool is the following theorem of Steinberg, [43], generalizing a theorem of Dolph, McLeod and Thoe, [13], for the case of Hilbert-Schmidt operators.

Theorem 2.4. Let $\mathcal{B}$ be a Banach space, and let $T(z)$ be an analytic family of compact operators defined for $z \in \Omega \subset \mathbf{C}$. Then either $I - T(z)$ is nowhere invertible for $z \in \Omega$ or $(I - T(z))^{-1}$ is meromorphic for $z \in \Omega$.

Let us write the Gel'fand-Levitan equation (2.3) in the symbolic form

$$(I + T(x;t))K(x,y;t) + F(x,y;t) = 0 , \tag{2.15}$$

where $T(x,t)$ denotes the family of integral operators

$$T(x,t)f(y) = \int_x^\infty f(z)H(z,y;t)dz . \tag{2.16}$$

It will always be assumed that $T(x,t)$ is a compact operator for each fixed (x,t). For instance, this is guaranteed if

$$\int_x^\infty \int_x^\infty |H(y,z;t)|^2 dydz < \infty ,$$

indeed, in this case T is Hilbert-Schmidt.

To apply Steinberg's theorem, we treat the time t as the complex parameter. (Note that it would not do any good to look at x as this parameter since the domain of integration for $T(x,t)$ depends on x, and so the operators could not possibly be analytic for a large enough class of functions.) Now, for all x, y, if the kernel $H(x,y;t)$ depends analytically on t for $t \in \Omega$, then the operators $T(x,t)$ depend analytically on t. If furthermore $F(x,y;t)$ is analytic in t, then Steinberg's theorem

implies that

$$K(x,y;t) = -(I + T(x,t))^{-1}F(x,y;t)$$

is, for each fixed (x,y), a meromorphic function of t. (It is one of the assumptions of complete integrability that the inverse exists for at least one t.) Therefore

$$Q[u(x,t)] = P[K(x,x;t)]$$

is also a meromorphic function of t for each fixed x.

Theorem 2.5. *If a system of partial differential equations is Q-completely integrable in the Banach space* $\mathcal{B}$, *and if the initial data* $u(x,0) \in \mathcal{B}$, *then the function* $Q[u(x,t)]$ *is meromorphic in* t *for* $t \in \Omega$ *and each fixed* x.

A slight generalization of this theorem will prove to be of use in the sequel. Suppose that the time axis is "skewed", by making the change of variables

$$(\tilde{x},\tilde{t}) = (x + \varepsilon t,t)$$

for some real ε. If $u = f(x,t)$ is the solution to the "unskewed" equation, then $\tilde{u} = \tilde{f}(\tilde{x},\tilde{t}) = f(\tilde{x} - \varepsilon\tilde{t},\tilde{t})$ is the solution in terms of the new coordinates. If we let

$$\tilde{K}(\tilde{x},\tilde{y};\tilde{t}) = K(\tilde{x} - \varepsilon\tilde{t},\tilde{y} - \varepsilon\tilde{t};\tilde{t}) ,$$

then $\tilde{K}$ is a solution of a Gel'fand-Levitan equation of the form

$$K(\tilde{x},\tilde{y};\tilde{t}) + F(\tilde{x} - \varepsilon\tilde{t},\tilde{y} - \varepsilon\tilde{t};\tilde{t}) + \int_{\tilde{x}}^{\infty} \tilde{K}(\tilde{x},\tilde{z};\tilde{t})H(\tilde{z} - \varepsilon\tilde{t},\tilde{y} - \varepsilon\tilde{t};\tilde{t})d\tilde{z} = 0 .$$

Therefore the "skewed" equation is also completely integrable, which gives the following theorem.

Theorem 2.6. *If a system of partial differential equations is Q-completely integrable in the Banach space* $\mathcal{B}$, *and if the initial data are in* $\mathcal{B}$, *then the function* $Q[u(x,t)]$ *is meromorphic in* (x,t) *for* $x \in \mathbf{C}$, $t \in \Omega$.

3. Symmetry Groups and Group-Invariant Solutions

Given a system of partial differential equations

$$\Delta(x,u) = 0, \quad x \in R^m, \quad u \in R^n , \tag{3.1}$$

a symmetry group will be a local Lie group of transformations G acting on the space $R^m \times R^n$ of independent and dependent variables which preserves the set of solutions of (3.1). The group acts on the solutions by transforming their graphs. For simplicity, we will restrict our attention to projectable groups, meaning those in which the transformations are all of the form $(\tilde{x},\tilde{u}) = (\alpha(x),\beta(x,u))$. If $u = f(x)$ is a solution, the transformed solution will be given by the formula

$$\tilde{u} = \tilde{f}(\tilde{x}) = \beta(\alpha^{-1}(\tilde{x}), f(\alpha^{-1}(\tilde{x}))) ,$$

provided α is invertible. The details of this theory can be found in [8], [34] and [37].

If G is a local, projectable, one-parameter group, its infinitesimal generator is a vector field of the form

$$\vec{v} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} \varphi^j(x,u) \frac{\partial}{\partial u^j} .$$

Denote the partial derivatives of the u^j by

$$u_k^j = \frac{\partial^{|k|} u^j}{\partial x^k} , \quad k = (k_1,\dots,k_m) ,$$

and let

$$D_i = \frac{\partial}{\partial x^i} + \sum_{j,k} u_{k,i}^j \frac{\partial}{\partial u_k^j} ,$$

where $k,i = (k_1,\dots,k_i + 1,\dots,k_m)$, denote the total derivative with respect to x^i. Define the prolongation of $\vec{v}$ to be the vector field

$$\mathrm{pr}\, \vec{v} = \vec{v} + \sum_{j,k} \varphi_k^j \frac{\partial}{\partial u_k^j} , \tag{3.2}$$

where

$$\varphi_k^j = D^k[\varphi^j - \sum_i \xi^i \frac{\partial u^j}{\partial x^i}] + \sum_i \xi^i u_{k,i}^j .$$

(Here $D^k = D_1^{k_1} D_2^{k_2} \ldots D_m^{k_m}$.) Then the following theorem provides the infinitesimal criterion for G to be a symmetry group of a given partial differential equation.

Theorem 3.1. *Let G be a connected local Lie group. Then in general G is a symmetry group of the system of partial differential equations (3.1) if and only if*

$$\mathrm{pr}\,\vec{v}(\Delta) = 0 \tag{3.3}$$

whenever $\Delta = 0$ for all infinitesimal generators $\vec{v}$ of G.

A precise statement and proof of this may be found in [34], or, in simplified form, in [35].

Example 3.2. Consider the generalized KdV equation

$$\Delta(x,u) \equiv u_t + u^p u_x + u_{xxx} = 0 \ , \tag{3.4}$$

where p is a positive integer. Let G denote the one-parameter group of scale transformations

$$(x,t,u) \to (e^{-p\lambda}x, e^{-3p\lambda}t, e^{2\lambda}u), \qquad \lambda \in \mathbb{R} \ .$$

The infinitesimal generator of G is

$$\vec{v} = -px\frac{\partial}{\partial x} - 3pt\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} \ .$$

From the prolongation formula (3.2), we find that

$$\mathrm{pr}\,\vec{v} = \vec{v} + (p+2)u_x\frac{\partial}{\partial u_x} + (3p+2)u_t\frac{\partial}{\partial u_t} + (2p+2)u_{xx}\frac{\partial}{\partial u_{xx}} + (3p+2)\frac{\partial}{\partial u_{xxx}} + \ldots \ .$$

Therefore

$$\mathrm{pr}\,\vec{v}[\Delta] = (3p+2)u_t + (3p+2)u^p u_x + (3p+2)u_{xxx} = (3p+2)\Delta \ ,$$

so that the infinitesimal criterion (3.3) is verified. We conclude that if $u = f(x,t)$ is any solution of (3.4), so also is $\tilde{u} = e^{2\lambda}f(e^{p\lambda}x, e^{3p\lambda}t)$. This may be checked directly.

In general, to find the symmetry group of a given system of partial differential equations, one looks for all vector fields $\vec{v}$ satisfying (3.3). This necessitates the solution of a large number of elementary partial differential equations for the coefficient functions ξ^i, φ^j of $\vec{v}$, which is easily done in practice, as in the above mentioned references.

Now, given a symmetry group G, a G-invariant (or self-similar) solution of (3.1) is a solution which is unchanged by the transformations in G. The fundamental property of G-invariant solutions is that, roughly speaking, they may all be found via the integration of a system of partial differential equations in fewer independent variables. To make this precise, we must assume that G acts "regularly" in the sense of Palais, [39]. This requires

i) that all the orbits of G have the same dimension, ι,

ii) that, for any point (x,u), there exist arbitrarily small neighborhoods N such that the intersection of any orbit O of G with N is a pathwise connected subset of O.

(The prototypical group actions excluded by the second requirement are the irrational flows on the torus.)

Under these two assumptions, it is well known that the quotient space $M = R^m \times R^n/G$, whose points correspond to the orbits of G, can be naturally endowed with the structure of a smooth (although not always Hausdorff) manifold. Moreover the G-invariant solutions of (3.1) are all obtained by integrating a reduced system $\Delta/G = 0$ of partial differential equations on M, which necessarily has ι fewer independent variables. Precise statements and proofs of these results may be found in [34].

For our purposes, the construction of the reduced system for the G-invariant solutions proceeds as follows: Local coordinate systems on the quotient manifold M are provided by a "complete set of functionally independent invariants of G", cf. [37]. If G is projectable, these are functions of the form

$$\xi^1(x),\dots,\xi^{m-\iota}(x),\ w^1(x,u),\dots,w^n(x,u)\ ,$$

which are unchanged under the action of G. The functional independence means that the Jacobian matrix

$$\begin{pmatrix} \partial\xi/\partial x & 0 \\ \partial w/\partial x & \partial w/\partial u \end{pmatrix}$$

is everywhere nonsingular. The reduced system $\Delta/G = 0$ will then be found in terms of the new independent variables ξ^i and the new dependent variables w^j.

Example 3.3. Return to the generalized KdV equation and the group of scale transformations G of the previous example. Note that G acts regularly on $Z_o = R^3 \sim \{0\}$, the origin being a singular point. The quotient manifold $M = Z_o/G$ is a complicated, non-Hausdorff manifold. However, if we restrict our attention to the subset defined by $t > 0$, then a complete set of functionally independent invariants of G is provided by

$$\xi = xt^{-1/3}, \qquad w = t^{2/3p}u \ . \tag{3.5}$$

Viewing w as a function of ξ, we compute

$$u = t^{-2/3p}w \ ,$$
$$u_t = -\frac{2}{3p}\, t^{-1-\frac{2}{3p}}\, w - \frac{1}{3}\, xt^{-\frac{4}{3}-\frac{2}{3p}}\, w' \ ,$$
$$u_x = t^{-\frac{1}{3}-\frac{2}{3p}}\, w' \ ,$$
$$u_{xxx} = t^{-1-\frac{2}{3p}}\, w''' \ ,$$

the primes denoting derivatives with respect to ξ. Substituting these expressions into (3.4), and factoring out $t^{-1-\frac{2}{3p}}$, we obtain the reduced ordinary differential equation

$$-\frac{2}{3p}\, w - \frac{1}{3}\, \xi w' + w^p w' + w''' = 0 \ . \tag{3.6}$$

Every solution of (3.6) gives rise to a scale-invariant solution of the generalized KdV equation, via the transformation (3.5). In the special case $p = 2$ (the modified KdV equation), (3.6) can be integrated once, so that

$$w'' = -\frac{1}{3}\, w^3 + \frac{1}{3}\, \xi w + k$$

for some constant k. This may be recognized as the second Painlevé transcendent, [23].

Now we restrict our attention to a Q-completely integrable system, $\Delta = 0$, of partial differential equations in two independent variables, (x,t). Let G be a one-parameter local projectable symmetry group of the given system, such that the transformations in G, when extended to complex values of the variables (x,t,u), are analytic. Let G_o denote the projected group action on (x,t)-space. Assume further that the action of G_o

on some domain D_o is regular in the sense of Palais, so that all the G-invariant solutions of $\Delta = 0$ defined over D_o are found by integrating a system of ordinary differential equations, $\Delta/G = 0$, defined over the image M_o of D_o in the quotient manifold M.

<u>Theorem 3.4</u>. <u>Suppose</u> $\Delta = 0$ <u>is a Ω-completely integrable system of partial differential equations in the Banach space</u> $\mathcal{B}$ <u>with an analytic, regular, projectable, one-parameter symmetry group</u> G. <u>If</u> $u = f(x,t)$ <u>is a G-invariant solution of</u> $\Delta = 0$ <u>with initial data lying in</u> $\mathcal{B}$, <u>then the combination corresponding to</u> Ω <u>of the solution of the reduced system of ordinary differential equations is meromorphic in</u> M_o, <u>the image of</u> $\mathbf{C} \times \Omega$ <u>in</u> M.

<u>Proof</u>. Since G_o is analytic, the orbits of G_o in the (x,t)-plane must be analytic curves. If the solution of the reduced equation had a singularity other than a pole on M_o, the corresponding G-invariant solution would have a similar singularity along the orbit corresponding to the singular point. This, however, would contradict Theorem 2.6.

Thus Theorem 3.4, in a certain restricted sense, states that the reduced equation for the G-invariant solutions must be of Painlevé type. However, since the initial data for the G-invariant solutions must lie in $\mathcal{B}$, it is not for every solution of the reduced equation that Ω is required to have only poles for singularities. In effect we can consider only those solutions which either decay sufficiently rapidly at $\pm \infty$ along the real axis, or are periodic along the real axis. This restriction seems inescapable given the particular method of proof. It would be extremely interesting to remove these restrictions and prove the conjecture of the introduction in fully generality.

4. Applications

I. The Generalized KdV Equations

Consider the equation

$$u_t + u^p u_x + u_{xxx} = 0 \ , \tag{4.1}$$

where p is a nonnegative integer. In Example 3.3, the equation governing the scale-invariant solutions was derived. However, this third order ordinary differential equation is rather complex to analyze in full, and we therefore apply our results to a simpler class of self-similar solutions, namely the travelling wave solutions. Here the symmetry group is

$$G_c : (x,t,u) \to (x + c\lambda, t + \lambda, u), \qquad \lambda \in R \ ,$$

where c denotes the velocity of the wave. The invariants of G_c are $\xi = x - ct, u,$ and the reduced equation for G_c-invariant solutions takes the form

$$u''' + u^p u' - cu' = 0 \ ,$$

primes denoting derivatives with respect to ξ. This can be integrated once:

$$u'' = \frac{-1}{p+1} u^{p+1} + cu + \frac{1}{2} d \ .$$

Multiplying by u', a further integration yields

$$(u')^2 = \frac{-2}{(p+1)(p+2)} u^{p+2} + cu^2 + du + e \ , \tag{4.2}$$

for some constants d,e. Thus the general travelling wave solution will be expressed in terms of the hyperelliptic function corresponding to the square root of the $(p + 2)$-th order polynomial on the right of (4.2). The following two results characterize the singularities of the solutions of (4.2).

<u>Theorem 4.1</u> <u>(Painlevé's Theorem)</u>. <u>Consider the ordinary differential equation</u>

$$G(u', u, \xi) = 0 \ ,$$

<u>where</u> G <u>is a polynomial in</u> u' <u>and</u> u, <u>and analytic in</u> ξ. <u>Then the movable singularities of the solutions are poles and/or algebraic branch-points</u>.

<u>Theorem 4.2</u>. <u>Consider the equation</u>

$$(u')^2 = R(u) \ , \tag{4.3}$$

<u>where</u> R <u>is a rational function of</u> u. <u>Then the solutions of (4.3) are all meromorphic in</u> $\mathbb{C}$ <u>if and only if</u> R <u>is a polynomial of degree not exceeding 4</u>.

The proofs may be found in Ince, [23], and Hille, [22, p. 683]. Note that if u has an algebraic branch point, so also does any linear combination of u and its derivatives. Therefore, for (4.1) to be linearly completely integrable, (4.2) must satisfy Theorem 4.2. Thus $p = 0,1,$ or $2,$ and in these cases the solutions are given by elliptic or trigonometric functions. Note that $p = 0$ corresponds to the linear case, $p = 1$ to the KdV equation, and $p = 2$ to the modified KdV equation, all of which are known to be integrable by inverse scattering.

To complete the demonstration that the generalized KdV equations are not linearly completely integrable for $p \neq 0,1,2,$ we must place the complete integrability in a suitable Banach space $\mathcal{B}$, and to do so we check the asymptotic behavior of the travelling wave solutions at $\pm\infty$. If we require that $u, u_x \to 0$ as $|x| \to \infty$, then $d = e = 0$ in (4.2). Moreover the polynomial on the right of (4.2) now has a double zero at $u = 0$ and a simple zero at $u_o = [\frac{1}{2}(p+1)(p+2)c]^{1/p}$. Standard techniques, cf. [47], allow us to conclude the existence of travelling wave solutions with positive velocities decaying exponentially for $|x| \to \infty$, and reaching an extreme value of u_o. Thus for p odd, the travelling waves are humps with u_o the peak value, while for p even, both humps and troughs occur. The important point, however, is the exponential decay of these waves for $|x| \to \infty$, and the fact that for $p \neq 0,1,2,$ they have complex nonpolar singularities. If therefore we take for the Banach space $\mathcal{B}$ a space of functions vanishing exponentially, we have shown that the generalized KdV equations are not linearly completely integrable in $\mathcal{B}$ for $p \neq 0,1,2,$ and this completes the demonstration that these equations can be solved by inverse scattering only when $p = 0,1$ or 2. This result is in accordance with numerical evidence, [15], that only in these special cases do the equations have soliton solutions.

II. <u>Nonlinear Klein-Gordon Equations</u>

Consider the nonlinear Klein-Gordon equation in characteristic coordinates

$$u_{xt} = f'(u) , \tag{4.4}$$

where f is an analytic function of u, real for real u, and prime denotes derivative. The cases we will be most interested in are when f is a polynomial or a

finite sum of exponential functions. We will determine necessary conditions on f for (4.4) to be linearly completely integrable by analysis of the singularities of the travelling wave solutions. If c is the velocity, $\xi = x - ct$, then the reduced equation for the G_c-invariant solutions of (4.4) is

$$-cu'' = f'(u) \ . \tag{4.5}$$

Multiplying (4.5) by u' and integrating yields

$$-\frac{c}{2}(u')^2 = f(u) + k \tag{4.6}$$

for some constant k. For simplicity we shall assume that k can be chosen so that u_1 (real) is a simple or double zero of $f(u) + k$ and there is a second consecutive simple or double zero for some real u_2. This assumption ensures that the initial data $u(x,0)$ can be chosen to lie in a suitable Banach space $\mathcal{B}$:

i) if u_1 and u_2 are simple zeros, so that a solution of (4.6) oscillates between u_1 and u_2, we take $\mathcal{B}$ to be a space of periodic functions;

ii) if u_1 is a double and u_2 a simple zero, so that a solution of (4.6) decays exponentially to u_1 as $|\xi| \to \infty$, we take $\mathcal{B}$ to be a space of functions exponentially converging;

iii) if u_1 and u_2 are double zeros, so that a solution of (4.6) tends exponentially to u_1 as $\xi \to \infty$ and to u_2 as $\xi \to -\infty$ (or vice versa), we can again take $\mathcal{B}$ to be a space of functions exponentially converging, but to different limits.

The following theorem (stated in the context of (4.4) although it applies generally) is an immediate consequence of considering a linear combination of u and its derivatives. It tells us what singularities are possible for solutions of linearly completely integrable equations.

Theorem 4.3. Suppose for some constant k that the analytic function $f(u) + k$ has two consecutive simple and/or double zeros on the real axis. Then, if the nonlinear Klein-Gordon equation (4.4) is linearly completely integrable in the relevant Banach space indicated above, it must be the case that any solution of (4.6) (with c having the opposite sign to $f(u) + k$ between the zeros) has as singularities only poles or logarithmic branch-points.

A logarithmic branch-point is by definition a singularity such that some linear combination of derivatives has a pole. It arises in practice if the scattering operator L depends only on $u_x, u_{xx}, \ldots,$ so that $Q[u]$ in turn depends only on derivatives, and to demonstrate that this situation can indeed rise, consider the sine-Gordon equation

$$u_{xt} = \sin u .$$

It was indicated in Example 2.3 that this is completely integrable, and to examine it in the context of Theorem 4.3 we take

$$f(u) = -\cos u, \qquad k = 0 .$$

The solution of (4.6) is then

$$\sqrt{2}\,\sin(\tfrac{1}{2}u) = \mathrm{sn}\{c^{-1/2}(\xi + \delta)\} ,$$

where sn is the Jacobi elliptic function with modulus $k = 1/\sqrt{2}$, cf. [9]. This is well defined for $c > 0$. Now sn has simple poles on a certain rectangular lattice in $\mathbb{C}$, and so u has logarithmic singularities at these lattice points. The reason for the appearance of these nonpolar singularities is the fact that u_x rather than u appears in the scattering operator L. We note that u_x on the other hand does have only poles for singularities.

Theorem 4.4. Suppose that $f(u)$ is a rational function, real for real u and such that, for some $k, f(u) + k$ has two consecutive simple and/or double zeros on the real axis. If the Klein-Gordon equation $u_{xt} = f'(u)$ is linearly completely integrable, then f is a polynomial of degree not exceeding 4.

The proof is immediate from Theorems 4.1-2.

To discuss the case where f is a polynomial of degree ≤ 4, one can try other similarity solutions of (4.4), or else quite different tests. For example, it can be shown, [12], that when f is of degree > 2, so that f' is nonlinear, (4.4) has only finitely many polynomial conservation laws, while a theorem of Gel'fand and Dikii, [17], [18], states that if a system of partial differential equations has a Lax representation, then there are an infinite number of polynomial conservation laws.

Next we consider the case where f is a finite sum of exponential functions

$$f(u) = \sum_{j=0}^{m} c_j e^{\alpha_j u}, \quad c_j, \alpha_j \in \mathbf{C} .$$

For simplicity, we restrict our attention to the case where $\alpha_j = n_j \alpha$ for some $\alpha \in \mathbf{C}$ and some rational numbers n_j. By dividing α by the common denominator of the n_j, we may assume the n_j are integers. Now let $v = \exp(\alpha u)$, so that $v' = \alpha v u'$. Thus v satisfies

$$-\frac{c}{2\alpha^2}(v')^2 = \Sigma\, c_j v^{n_j+2} . \tag{4.7}$$

Note that Theorem 4.2 cannot be applied here since v may have singularities not shared by u. However, since $u' = v'/\alpha v$, it is necessary to find conditions on (4.7) such that the function v'/v, for solutions v, has no movable algebraic branch-points. This requires a more detailed investigation of the proof of Theorem 4.2. It suffices for our purposes to note the following:

<u>Lemma 4.5</u>. <u>Consider the ordinary differential equation</u>

$$(v')^2 = v^{-n} P(v) ,$$

<u>where</u> P <u>is a polynomial with</u> $P(0) \neq 0$ <u>and</u> n <u>is a positive integer. Then for any</u> $\xi_o \in \mathbf{C}$ <u>there is a solution</u> v <u>with algebraic branch-point at</u> ξ_o. <u>This solution has a Puiseux expansion</u>

$$v(\xi) = \sum_{j=1}^{\infty} a_j (\xi - \xi_o)^{jr}$$

<u>with</u> $a_1 \neq 0$, <u>and the rational number</u> r <u>is given by</u>

i) $r = (m+1)^{-1}$ <u>if</u> $n = 2m$,

ii) $r = 2(2m+3)^{-1}$ <u>if</u> $n = 2m+1$.

The proof of this result can be inferred from Hille, [22, pp. 681-682].

Lemma 4.6. Suppose v has an algebraic branch-point at ξ_o. Then v'/v has no branch-point at ξ_o if and only if $v(\xi) = (\xi - \xi_o)^r f(\xi)$ for r rational and f meromorphic at ξ_o.

Proof. Assume without loss of generality that $\xi_o = 0$. Let v have the Puiseux expansion

$$v(\xi) = \xi^{mr} \sum_{j=0}^{\infty} a_j \xi^{jr} ,$$

where m is an integer and $a_o \neq 0$. Let a_k be the first nonzero coefficient for which kr is not an integer, if such exists. Now

$$\frac{1}{v} = \xi^{-mr} \sum_{j=0}^{\infty} b_j \xi^{jr} ,$$

where $b_o = a_o^{-1}$ and the first nonzero coefficient b_j with jr not an integer is $b_k = -a_k a_o^{-2}$. Furthermore

$$v' = \xi^{mr-1} \sum_{j=0}^{\infty} (m + j) r a_j \xi^{jr} .$$

Therefore

$$\frac{v'}{v} = \xi^{-1} \sum_{j=0}^{\infty} c_j \xi^{jr}$$

and the coefficient of ξ^{kr} is

$$c_k = -mra_k a_o^{-1} + (m + k) r a_k a_o^{-1} ,$$

which vanishes only when $a_k = 0$. This proves the lemma.

Proposition 4.7. Consider the ordinary differential equation

$$(v')^2 = \sum_{j=-n}^{N} b_j v^j . \tag{4.8}$$

Given $\xi_o \in \mathbf{C}$, there exists a solution v of (4.8) such that v'/v has an algebraic branch-point at ξ_o, unless (4.8) is of the special form

$$(v')^2 = \sum_{j=-2}^{2} c_j v^{jk+2} \tag{4.9}$$

for some integer k.

Proof. Let $\xi_o = 0$ and assume $b_N \neq 0$, $b_{-n} \neq 0$. By Lemma 4.6 all solutions must be of the form $v(\xi) = \xi^r f(\xi)$ with r rational and f meromorphic at 0 if we are to avoid an algebraic branch-point for v'/v. Thus

$$(v')^2 = \xi^{2r}(r\xi^{-1}f + f')^2 \ ,$$

and

$$v^j = \xi^{jr} f^j \ ,$$

so that, equating the fractional powers of ξ, we see that $b_j = 0$ unless $jr = 2r + \iota$ for some integer ι. If $n > 0$, it follows from Lemma 4.5 that $b_j = 0$ unless

i) $j \equiv 2 \bmod (m + 1)$ for $n = 2m$,

or ii) $2j \equiv 4 \bmod (2m + 3)$ for $n = 2m + 1$.

In particular, the only negative values of j which satisfy these congruences are $1 - \frac{1}{2}n$ and $-n$, the first value occurring only when n is even.

Next set $w = 1/v$. Then (4.8) becomes

$$(w')^2 = \sum_{j=-n}^{N} b_j w^{4-j} \ .$$

Since $w'/w = -v'/v$, w must satisfy the same conditions as v. Therefore, if $N > 4$, $b_j = 0$ unless

i) $j \equiv 2 \bmod (M - 1)$ if $N = 2M$,

or ii) $2j \equiv 4 \bmod (2M - 2)$ if $N = 2M + 1$.

The only positive values of j satisfying these are N, $\frac{1}{2}N + 1$ and 2, the second only if N is even. Comparison of the two sets of congruences then shows that (4.8) must be of the required form.

Theorem 4.8. Suppose $f(u)$ is a linear combination of exponential functions $e^{\alpha_j u}$ with $\alpha_j = n_j\alpha$, n_j rational, α complex. Suppose further that $f(u)$ is real for u real, and that, for some real k, $f(u) + k$ has two consecutive simple and/or double zeros on the real axis. If the Klein-Gordon equation $u_{xt} = f'(u)$ is linearly completely integrable, then f must be of the special form

$$f(u) = \sum_{j=-2}^{2} c_j e^{j\beta u} , \tag{4.10}$$

where β is a rational multiple of α.

It is interesting that the form (4.10) for f includes the double sine-Gordon equation

$$u_{xt} = a \sin\alpha u + b \sin(\tfrac{1}{2} \alpha u) ,$$

for which numerical studies of Dodd and Bullough, [11], indicate the existence of soliton solutions. A recent preprint by Gibbons and Fordy, [20], contains the result that the special case of (4.10) when $f(u) = e^{2u} + e^{-u}$ does have a Lax representation, but it is not known whether the result extends to a general function $f(u)$ of the form (4.10).

III. Model Wave Equations of Whitham and Benjamin

The integro-differential equation

$$u_t + uu_x + \mathcal{H}[u_x] = 0 , \tag{4.11}$$

where $\mathcal{H}$ is the integral operator

$$\mathcal{H}[f](x) = \int_{-\infty}^{\infty} H(x - y)f(y)dy ,$$

was proposed by Whitham, [46], [47], as an alternative to the KdV equation for long waves in shallow water which could also model breaking and peaking. Here H is taken to be the Fourier transform of the desired phase velocity $c(k)$, where k is the wave number. Of particular interest is the case

$$c(k) = \frac{1}{\nu^2 + k^2} , \qquad \nu > 0 ,$$

so that

$$H(x) = \frac{1}{2\nu} e^{-\nu|x|} .$$

Note that H is the Green's function of the operator $D^2 - \nu^2 = \mathcal{D}$ so that (4.11) is equivalent to the differential equation

$$\mathcal{D}[u_t + uu_x] + u_x = 0 . \qquad (4.12)$$

It can be shown, [15], that (4.12) possesses travelling wave solutions u, with $|u| \to 0$ as $|x| \to \infty$, and amplitudes between 0 and some maximum height. Computer studies indicate that these waves may be solitons, i.e. interact cleanly. One possibly undesirable feature of (4.11) is the extremely fast propagation of short-wave components, and for this reason Benjamin, Bona and Mahony, [7], proposed the alternative model

$$u_t + uu_x - \mathcal{H}[u_t] = 0 . \qquad (4.13)$$

Again, in the special case, (4.13) can be rewritten as

$$\mathcal{D}[u_t + uu_x] - u_t = 0 . \qquad (4.14)$$

In general, we will let $\mathcal{D}$ be any constant coefficient linear differential operator

$$\mathcal{D} = \sum_{i=0}^{n} c_i D^i, \qquad c_n \neq 0 .$$

We show here that the model equations (4.12), (4.14) cannot be integrable by inverse scattering methods. As usual, consider the travelling wave solutions of these equations. If c denotes the velocity, then the reduced equation, after integration, is

$$\mathcal{D}[\tfrac{1}{2}(u - c)^2] + a(u + d) = 0 . \qquad (4.15)$$

Here d is a constant of integration, $a = 1$ in the Whitham model, $a = c$ in the Benjamin model, and D now denotes $d/d\xi$, $\xi = x - ct$. Since n-th order equations of Painlevé type have not been classified, we resort to Painlevé's original "α-method" to analyze the singularities of the solutions of (4.15). The basic result is found in Ince, [23, p. 319].

Lemma 4.9. Suppose $\Delta(u,\xi,\alpha) = 0$ *is an analytically parametrized family of ordinary differential equations for* α *in some domain* Ω *containing* 0 *as an interior point. If*

the general solution $u(\xi,\alpha)$ _is uniform in_ ξ _for_ $\alpha \in \Omega \sim \{0\}$, _then it will be uniform for_ $\alpha = 0$.

In our case, let $\xi = \xi_0 + \alpha\zeta$. Then if we consider u as a function of ζ, (4.15) becomes

$$(c_n D^n + \alpha c_{n-1} D^{n-1} + \dots + \alpha^n c_0)[\tfrac{1}{2}(u - c)^2] + \alpha^n a(u + d) = 0 ,$$

where D now denotes $d/d\zeta$. For $\alpha = 0$, this reduces to

$$D^n(\tfrac{1}{2}(u - c)^2) = 0 ,$$

the solution of which is

$$u = c + \sqrt{P_n(\zeta)}$$

for an arbitrary polynomial P_n of degree $\leq n - 1$. This, for appropriate P_n, has an algebraic branch-point at $\zeta = 0$, so that, by the lemma, solutions of (4.15) must also have non-logarithmic branch-points. (This involves a slight extension of the lemma above, but it is easy to infer its truth from the proof given by Ince.) If these solutions also satisfy decay or periodicity properties, Theorem 3.4 (together with Theorem 4.3) shows that model equations (4.12), (4.14) cannot be linearly completely integrable. In particular, Whitham's equation with $\mathcal{D} = D^2 - \nu^2$ is not integrable by inverse scattering.

IV. The BBM Equation

The equation

$$u_t + uu_x - u_{xxt} = 0 \tag{4.16}$$

was proposed by Benjamin, Bona and Mahony, [7], as an alternative model to the KdV equation for the description of long waves in shallow water. In [36] it was shown to possess only three independent conservation laws, and therefore by the results of Gel'fand and Dikii cannot be completely integrable. Our consideration of this example runs into difficulties because the self-similar solutions do not satisfy any decay or periodicity properties, and the functions $\mathcal{D}$ we can allow are limited, but we will indicate the method here.

First we note that (4.16) admits the symmetry group

$$G : (x,t,u) \to (x, e^{-\lambda}t, e^{\lambda}u), \qquad \lambda \in \mathbf{R} ,$$

of scale transformations. Invariants of G are provided by x and $w = tu$, for $t > 0$, and the reduced equation for G-invariant solutions is then

$$w'' + ww' - w = 0 , \tag{4.17}$$

the primes denoting derivatives with respect to x. It can be readily checked, by the procedure in Ince, [23], that (4.17) is not of Painlevé type. Indeed, it is of type i(b) on page 330 of Ince. Applying the α-method as Ince does, one can readily check that branch-points appear, although possibly only logarithmic, and this, granted the existence of a suitable Banach space $\mathcal{B}$, would show that the BBM equation is not Q-completely integrable for Q, say, the identity.

However, a closer investigation of the behavior of the real solutions of (4.17) is required. Since x does not appear, it can be integrated to yield

$$(1 - w')e^{w'} = ce^{-\frac{1}{2}w^2} . \tag{4.18}$$

In principle, this equation can again be integrated by solving for w' in terms of w. To investigate the solutions qualitatively, note that $w' = 0$ if and only if $w^2 = 2 \log c$, $c > 1$. The only double root is when $c = 1$, and only in this case do solutions decay at $+\infty$ or $-\infty$. However, it is readily seen that a solution decaying at one endpoint cannot decay at the other, nor are periodic solutions possible. Thus we are unable to apply our results to this case.

V. Lax Pairs of Composite Order

Gel'fand and Dikii, [17], [18], succeeded in classifying all Lax pairs of differential operators of the following special type. Let

$$L_n = D^n + u_{n-2}D^{n-2} + \dots + u_1 D + u_0$$

be a scalar differential operator of order n with $u = (u_0,\dots,u_{n-2})$ independent C^∞ functions, and $D = d/dx$. They showed that for each integer m not a multiple of n, there is a differential operator

$$P_m = D^m + p_{m,m-2}D^{m-2} + \dots + p_{m,1}D + p_{m,0}$$

of order m, the $p_{m,i}$ being polynomials in the u_j and their derivatives, such that the Lax representation

$$\frac{\partial L_n}{\partial t} = [P_m, L_n]$$

is a genuine, nontrivial system of evolution equations

$$u_t = K_m(u) . \tag{4.19}$$

Moreover, the P_m are unique if we require the coefficients $p_{m,j}$ to have no constant term.

Consider the stationary solutions of the system (4.19), i.e. those in which u is independent of t. These satisfy the system $K_m(u) = 0$, or equivalently, the "stationary Lax representation"

$$[P_m, L_n] = 0 . \tag{4.20}$$

Theorem 4.10. If the orders n, m of the operators L_n, P_m in the Lax representation of (4.19) are not relatively prime integers, then stationary solutions of (4.19) with arbitrary singularities in the complex plane exist.

Proof. Let $k > 1$ be the greatest common divisor of m and n. Consider the operator

$$M_k = D^k + v_{k-2}D^{k-2} + \dots + v_1 D + v_0 ,$$

whose coefficients $v_j(x)$ are sufficiently differentiable for $x \in R$ but are otherwise arbitrary functions. Then

$$L_{n,0} = (M_k)^{n/k}, \quad P_{m,0} = (M_k)^{m/k}$$

obviously satisfy the stationary Lax representation (4.20) and, moreover, using the formalism of Gel'fand and Dikii, it is easy to prove that $P_{m,0}$ is derivable from $L_{n,0}$ via the same formulae as gave P_m from L_n. Therefore each such M_k gives a stationary solution of the evolutionary system (4.19).

Now suppose that L_n is any such operator, where n is a composite number. If there exists a Gel'fand-Levitan type of integral equation for solving the inverse problem for the operator L_n, then Theorem 3.4 would imply the meromorphic character of the group-invariant solutions of the evolutionary system (4.19), using similar arguments to those used in the integration of the Korteweg-de Vries equation. This, however, is in contradiction to Theorem 4.10 for the case of time-invariant solutions. (The relevant symmetry group is just translation in t.) This indicates that such a differential operator of composite order does not have an inverse-scattering formalism in the sense that

the Schrödinger operator does - either no such Gel'fand-Levitan equation exists, or the assumptions regarding analyticity are not justified. Indeed, we know of no such Gel'fand-Levitan equation for any operator of composite order, e.g. for order $n = 4$.

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A completely integrable partial differential equation is one which has a Lax representation, or, more precisely, can be solved via a linear integral equation of Gel'fand-Levitan type, the classic example being the Korteweg-de Vries equation. An ordinary differential equation is of Painlevé type if the only singularities of its solutions in the complex plane are poles. It is shown that, under certain restrictions, if G is an analytic, regular symmetry group of a completely integrable partial differential equation, then

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the reduced ordinary differential equation for the G-invariant solutions is necessarily of Painlevé type. This gives a useful necessary condition for complete integrability, which is applied to investigate the integrability of certain generalizations of the Korteweg-de Vries equation, Klein-Gordon equations, some model nonlinear wave equations of Whitham and Benjamin, and the BBM equation.

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